

The multi-mode quantum Entropy Power Inequality

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The quantum version of a fundamental entropic data-processing inequality is presented. It establishes a lower bound for the entropy which can be generated in the output channels of a scattering process which involves a collection of independent input bosonic modes (e.g. the modes of the electromagnetic field). The impact of this inequality in quantum information theory is potentially large and some relevant implications are considered in this work.

I. INTRODUCTION

Entropic inequalities are a fundamental tool in classical information and communication theory [1], where they can be used to bound the efficiency of data processing procedures. For this reason, a large effort has been devoted to this subject, with results such as the Entropy Power Inequality [2–7], used in the proof of a stronger version of the central limit theorem [8] and crucial in the computation of the capacities of various classical channels [9], and the Brunn-Minkowski inequality (for a review, see [10] or chapter 17 of [1]). For the same reason, entropic inequalities are fundamental also in the context of quantum information theory [11]. In particular the long-standing problem of determining the classical capacity of phase-insensitive quantum bosonic Gaussian channels [12, 13] was linked to a lower bound conjectured to hold for the minimum von Neumann entropy achievable at the output of a transmission line (the Minimum Output Entropy conjecture, MOE) [14]. While these issues were recently solved in Refs. [15–17] a stronger version of the MOE relation, arising from a suitable quantum generalization of the Entropy Power Inequality, is still not proved. This new relation, called Entropy Photon-number Inequality (EPnI) [18], turns out to be crucial in the determining the classical capacity regions of the quantum bosonic broadcast [19, 20] and wiretap [21] channels. A partial solution has been provided in [22] by proving a weaker version of the EPnI, called quantum Entropy Power Inequality (qEPI) and first introduced and studied by König and Smith in Ref. [23, 24]. Both the EPnI and the qEPI establish lower bounds on the entropy achievable in one of the output channels originating when two bosonic input modes, initialized in factorized input states of assigned entropies, are coupled via a beam-splitter or an amplifier transformation [25]. Here we present a multi-mode generalization of the qEPI which applies to the context where an arbitrary collection of independent input bosonic modes undergo to a scattering process which mixes them according to some linear coupling - see Fig. 1 for a schematic representation of the model. This new inequality permits to put bounds on the MOE inequality, still unproved for non gauge-covariant multi-mode channels, and then on the classical capacity

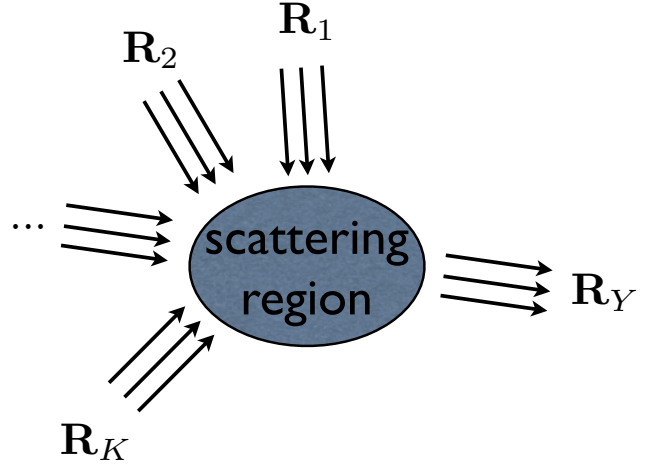


FIG. 1. (Color online): Graphical representation of the scheme underlying the multi-mode qEPI (10): it establishes a lower bound on the von Neumann entropy emerging from the output port indicated by R_Y of a multi-mode scattering process that linearly couples K independent sets of bosonic input modes (each containing n modes), initialized into factorized density matrices.

of any (not necessarily phase-insensitive) quantum Gaussian channel. Besides, our finding can find potential applications in extending the results of [22] on the classical capacity region of the quantum bosonic broadcast channel to the Multiple-Input Multiple-Output setting (see e.g. Ref. [26]), providing upper bounds for the associated capacity regions.

II. THE PROBLEM

The generalization of the qEPI we discuss in the present work finds a classical analogous in the multi-variable version of the EPI [2–7]. The latter applies to a set of K independent random variables \mathbf{X}_α , $\alpha = 1, \dots, K$, valued in \mathbb{R}^m and collectively denoted by \mathbf{X} , with factorized probability densities $p_X(\mathbf{x}) = p_1(\mathbf{x}_1) \dots p_K(\mathbf{x}_K)$, and with Shannon differential en-

tropies [2] $H_\alpha = -\langle \ln p_\alpha(\mathbf{x}_\alpha) \rangle$ (the $\langle \dots \rangle$ representing the average with respect to the associated probability distribution). Defining hence the linear combination

$$\mathbf{Y} = M \mathbf{X} = \sum_{\alpha=1}^K M_\alpha \mathbf{X}_\alpha, \quad (1)$$

where M is an $m \times Km$ real matrix made by the K blocks M_α , each of dimension $m \times m$, the multi-variable EPI gives an (optimal) lower bound to the Shannon entropy H_Y of \mathbf{Y}

$$\exp[2H_Y/m] \geq \sum_{\alpha=1}^K |\det M_\alpha|^{\frac{2}{m}} \exp[2H_\alpha/m]. \quad (2)$$

In the original derivation [2–7] this inequality is proved under the assumption that all the M_α coincide with the identity matrix, i.e. for $\mathbf{Y} = \sum_{\alpha=1}^K \tilde{\mathbf{X}}_\alpha$. From this however Eq. (2) can be easily established choosing $\tilde{\mathbf{X}}_\alpha = M_\alpha \mathbf{X}_\alpha$, and remembering that the entropy \tilde{H}_α of $\tilde{\mathbf{X}}_\alpha$ satisfies $\tilde{H}_\alpha = H_\alpha + \ln |\det M_\alpha|$. It is also worth observing that for Gaussian variables the exponentials of the entropies H_α and H_Y are proportional to the determinant of the corresponding covariance matrices, i.e.

$$H_\alpha = \frac{1}{2} \ln \det (\pi e \sigma_\alpha)$$

and

$$H_Y = \frac{1}{2} \ln \det (\pi e \sigma_Y),$$

with

$$\sigma_\alpha = 2 \langle \Delta \mathbf{x}_\alpha \Delta \mathbf{x}_\alpha^T \rangle, \quad \sigma_Y = 2 \langle \Delta \mathbf{y} \Delta \mathbf{y}^T \rangle$$

and

$$\Delta \mathbf{x}_\alpha = \mathbf{x}_\alpha - \langle \mathbf{x}_\alpha \rangle, \quad \Delta \mathbf{y} = \mathbf{y} - \langle \mathbf{y} \rangle.$$

Accordingly in this special case Eq. (2) can be seen as an instance of the Minkowski's determinant inequality [27] applied to the identity

$$\sigma_Y = \sum_{\alpha=1}^K M_\alpha \sigma_\alpha M_\alpha^T, \quad (3)$$

and it saturates under the assumption that the matrices entering the sum are all proportional to a given matrix σ , i.e.

$$M_\alpha \sigma_\alpha M_\alpha^T = c_\alpha \sigma, \quad (4)$$

with c_α being arbitrary (real) coefficients.

In the quantum setting the random variables get replaced by $n = \frac{m}{2}$ bosonic modes (for each mode there are two quadratures, Q and P), and instead of probability distributions over \mathbb{R}^{2n} , we have the quantum density matrices $\hat{\rho}_\alpha$ on the Hilbert space $L^2(\mathbb{R}^n)$. For each α , let

$\hat{\mathbf{R}}_\alpha$ be the column vector that collectively denotes all the quadratures of the α -th subsystem:

$$\hat{\mathbf{R}}_\alpha = \left(\hat{Q}_\alpha^1, \hat{P}_\alpha^1, \dots, \hat{Q}_\alpha^n, \hat{P}_\alpha^n \right)^T, \quad \alpha = 1, \dots, K. \quad (5)$$

The $\hat{\mathbf{R}}_\alpha$ satisfy the canonical commutation relations

$$[\hat{\mathbf{R}}_\alpha, \hat{\mathbf{R}}_\beta^T] = \delta_{\alpha\beta} \Delta \hat{\mathbf{1}}, \quad (6)$$

where Δ is the symplectic matrix (see e.g. [28]) given by

$$\Delta = \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider then totally factorized input states $\hat{\rho}_X = \bigotimes_{\alpha=1}^K \hat{\rho}_\alpha$, where $\hat{\rho}_\alpha$ is the density matrix of the α -th input. The analogue of (1) is defined with

$$\hat{\rho}_Y = \Phi(\hat{\rho}_X) = \text{Tr}_Z \left(\hat{U} \hat{\rho}_X \hat{U}^\dagger \right), \quad (7)$$

where $\hat{U} : \mathcal{H}_X \rightarrow \mathcal{H}_Y \otimes \mathcal{H}_Z$ is an isometry between the input Hilbert space \mathcal{H}_X and the tensor product of the output Hilbert space \mathcal{H}_Y with an ancilla Hilbert space \mathcal{H}_Z , satisfying

$$\hat{U}^\dagger \hat{\mathbf{R}}_Y \hat{U} = M \hat{\mathbf{R}}_X = \sum_{\alpha=1}^K M_\alpha \hat{\mathbf{R}}_\alpha. \quad (8)$$

As before, M is a $2n \times 2Kn$ real matrix made by the $2n \times 2n$ square blocks M_α . The canonical commutation relations (6) on $\hat{\mathbf{R}}_Y$ together with the unitarity of \hat{U} impose the constraint

$$\sum_{\alpha=1}^K M_\alpha \Delta M_\alpha^T = \Delta.$$

Notice that at the level of the covariance matrices Eq. (8) induces the same mapping (3) that holds in the classical scenario (in this case however one has

$$\sigma_\alpha := \left\langle \left\{ \hat{\mathbf{R}}_\alpha - \langle \hat{\mathbf{R}}_\alpha \rangle, \hat{\mathbf{R}}_\alpha^T - \langle \hat{\mathbf{R}}_\alpha^T \rangle \right\} \right\rangle,$$

$$\sigma_Y := \left\langle \left\{ \hat{\mathbf{R}}_Y - \langle \hat{\mathbf{R}}_Y \rangle, \hat{\mathbf{R}}_Y^T - \langle \hat{\mathbf{R}}_Y^T \rangle \right\} \right\rangle$$

with $\langle \dots \rangle = \text{Tr}[\hat{\rho}_X \dots]$ and $\{\dots, \dots\}$ representing the anti-commutator). The isometry \hat{U} in (7) does not necessarily conserve energy, i.e. it can contain active elements, so that even if the input $\hat{\rho}_X$ is the vacuum on all its K modes, the output $\hat{\rho}_Y$ can be thermal with a nonzero temperature. For $K = 2$, the beam-splitter of parameter $0 \leq \lambda \leq 1$ is easily recovered with

$$M_1 = \sqrt{\lambda} \mathbb{1}_{2n}, \quad M_2 = \sqrt{1-\lambda} \mathbb{1}_{2n}.$$

To get the quantum amplifier of parameter $\kappa \geq 1$, we must take instead

$$M_1 = \sqrt{\kappa} \mathbb{1}_{2n}, \quad M_2 = \sqrt{\kappa - 1} T_{2n},$$

where T_{2n} is the n -mode time-reversal

$$T_{2n} = \bigoplus_{k=1}^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can now state the multi-mode qEPI: the von Neumann entropies $S_\alpha = -\text{Tr}(\hat{\rho}_\alpha \ln \hat{\rho}_\alpha)$ satisfy the analogue of (2)

$$\exp[S_Y/n] \geq \sum_{\alpha=1}^K \lambda_\alpha \exp[S_\alpha/n], \quad (9)$$

where we have defined $\lambda_\alpha := |\det M_\alpha|^{\frac{1}{n}}$. The qEPI (9) was proved [22, 23] only in the simple cases of the quantum beam-splitter and amplifier. As already noticed, in the classical setting the generalized inequality (2) is a trivial consequence of the case with all the M_α equal to the identity. In the quantum setting this is not the case and one needs to find a proof that works for all possible choices of the M_α . The main result of the present paper is exactly to tackle this problem.

III. THE PROOF

The proof of Eq. (9), even if with some non trivial modifications, proceeds along the same line of the one in [23] and [22]. Specifically, inspired from what we know about the classical case, we expect that the qEPI should be saturated by quantum Gaussian states [13, 28] with high entropy and whose covariance matrices σ_α fulfill the condition (4) (the high entropy limit being necessary to ensure that the associated quantum Gaussian states behave as classical Gaussian probability distributions). Suppose hence we do apply a transformation on the input modes of the system which depends on a real parameter τ that plays the role of an effective temporal coordinate, and which is constructed in such a way that, starting from $\tau = 0$ from the input state $\hat{\rho}_X$ it will drive the modes towards such optimal Gaussian configurations in the asymptotic limit $\tau \rightarrow \infty$ — see Sec. III A. Accordingly for each $\tau \geq 0$ we will have an associated value for the entropies S_α and S_Y which, if the qEPI is correct, should still fulfill the bound (9). To verify this it is useful to put the qEPI (9) in the rate form

$$\frac{\sum_{\alpha=1}^K \lambda_\alpha \exp[S_\alpha/n]}{\exp[S_Y/n]} \leq 1. \quad (10)$$

We will then study the left-hand-side of Eq. (10) showing that its parametric derivative is always positive (see Sec. III B) and that that for $\tau \rightarrow \infty$ it tends to 1 (see section III C).

A. Parametric evolution

In this section we find the parametric evolution suitable to the proof. For this purpose for each input mode α we enforce the following dynamical process

$$\frac{d}{dt} \hat{\rho}_\alpha(t) = \mathcal{L}_{\gamma_\alpha}(\hat{\rho}_\alpha(t)), \quad (11)$$

characterized by the Lindblad super-operator

$$\mathcal{L}_{\gamma_\alpha}(\hat{\rho}) := -\frac{1}{4} \left[\left(\Delta^{-1} \hat{\mathbf{R}}_\alpha \right)^T, \gamma_\alpha \left[\Delta^{-1} \hat{\mathbf{R}}_\alpha, \hat{\rho} \right] \right], \quad (12)$$

where if M_α is invertible, $\gamma_\alpha := \lambda_\alpha M_\alpha^{-1} M_\alpha^{-T}$ is positive definite, and if M_α is not invertible, $\gamma_\alpha := 0$, i.e. we do not evolve the corresponding input. The generator (12) commutes with translations, i.e.

$$\mathcal{L}_{\gamma_\alpha} \left(\hat{D}_\mathbf{x} \hat{\rho} \hat{D}_\mathbf{x}^\dagger \right) = \hat{D}_\mathbf{x} \mathcal{L}_{\gamma_\alpha}(\hat{\rho}) \hat{D}_\mathbf{x}^\dagger, \quad (13)$$

where $\hat{D}_\mathbf{x} := \exp \left(i \mathbf{x}^T \Delta^{-1} \hat{\mathbf{R}}_\alpha \right)$ are the displacement operators of the system [28]. Furthermore it induces a diffusive evolution which adds Gaussian noise into the system driving it toward the set of Gaussian states while inducing a linear increase of the mode covariance matrix, i.e.

$$\frac{d}{dt} \sigma_\alpha(t) = \gamma_\alpha \implies \sigma_\alpha(t) = \sigma_\alpha(0) + t \gamma_\alpha, \quad (14)$$

which boosts its entropy. Notice that the choice we made on γ_α ensures that for large enough t , $M_\alpha \sigma_\alpha(t) M_\alpha^T$ will asymptotically approach the saturation condition (4) of the classical EPI with the matrix σ being the identity operator. We now let the various input modes evolve independently with their own processes (11) for different time intervals $t_\alpha \geq 0$: accordingly the input state of the system is mapped from $\hat{\rho}_X$ to $\hat{\rho}_X(t_1, t_2, \dots, t_K) = \bigotimes_{\alpha=1}^K \hat{\rho}_\alpha(t_\alpha)$ with $\hat{\rho}_\alpha(t_\alpha)$ being the evolved density matrix of the α -mode, its von Neumann entropy being $S_\alpha(t_\alpha)$. Next in order to get a one parameter trajectory we link the various time intervals t_α by parametrizing them in terms of an external coordinate $\tau \geq 0$ by enforcing the following constraint:

$$\frac{d}{d\tau} t_\alpha(\tau) = \exp[S_\alpha(t_\alpha(\tau))/n], \quad t_\alpha(0) = 0. \quad (15)$$

This is a first order differential equation which, independently from the particular functional dependence of $S_\alpha(t_\alpha)$, always admits a solution. Furthermore, since the right-hand-side of Eq. (15) is always greater than or equal to 1, it follows that $t_\alpha(\tau)$ diverges as τ increases, i.e.

$$\lim_{\tau \rightarrow \infty} t_\alpha(\tau) = \infty. \quad (16)$$

Accordingly in the asymptotic limit of large τ , the mapping $\hat{\rho}_X \rightarrow \hat{\rho}_X(\tau) = \bigotimes_{\alpha=1}^K \hat{\rho}_\alpha(t_\alpha(\tau))$ will drive the system toward the tensor product of the asymptotic points defined by the diffusive local master equation (11). As

we shall see in Sec. III C this implies that the rate on the left-hand-side of Eq. (10) will asymptotically reach the value 1. In order to evaluate this limit, as well as to study the parametric derivative in τ of such a rate, we need to compute the functional dependence upon τ of the von Neumann entropy S_Y of the output modes associated with the coordinates $\hat{\mathbf{R}}_Y$. It turns out that with the choice (15), the parametric evolution of the input mode induces a temporal evolution of the output modes which, expressed in terms of the local time coordinate t_Y having parametric dependence upon τ given by

$$t_Y(\tau) = \sum_{\alpha=1}^K \lambda_{\alpha} t_{\alpha}(\tau), \quad (17)$$

is still in the form of (11) with the operators $\hat{\mathbf{R}}_{\alpha}$ appearing in (12) being replaced by $\hat{\mathbf{R}}_Y$, and with the matrix γ_{α} being replaced by $\mathbb{1}_{2n}$. Accordingly in this case Eq. (14) becomes

$$\frac{d\sigma_Y}{dt_Y} = \mathbb{1}_{2n} \implies \sigma_Y(t_Y) = \sigma_Y(0) + t_Y \mathbb{1}_{2n}. \quad (18)$$

B. Evaluating the parametric derivative of the rate

Define the Fisher information matrix of a quantum state $\hat{\rho}$ as the Hessian with respect to \mathbf{x} of the relative entropy between $\hat{\rho}$ and $\hat{\rho}$ displaced by \mathbf{x} :

$$J_{ij}(\hat{\rho}) := \frac{\partial^2}{\partial x^i \partial x^j} S\left(\hat{\rho} \left\| \hat{D}_{\mathbf{x}} \hat{\rho} \hat{D}_{\mathbf{x}}^{\dagger} \right\|_{\mathbf{x}=0}\right). \quad (19)$$

An explicit computation shows

$$J = \text{Tr} \left(\left[\Delta^{-1} \hat{\mathbf{R}}, \left[\left(\Delta^{-1} \hat{\mathbf{R}} \right)^T, \hat{\rho} \right] \right] \ln \hat{\rho} \right). \quad (20)$$

The key observation is that the Fisher information matrix is easily related to the derivative of the entropy with respect to the evolution (12) through a generalization of the de Bruijn identity of [22, 23]:

$$\frac{d}{dt} S(\hat{\rho}(t)) = \frac{1}{4} \text{tr} \left(J(\hat{\rho}(t)) \frac{d}{dt} \sigma(t) \right). \quad (21)$$

With (21) and (15) we can compute the time derivatives of the entropies which enter in the definition of the rate in the left-hand-side of Eq. (10). Specifically from Eqs. (14), (15), (17) and (18) we get

$$\frac{dS_{\alpha}}{d\tau} = \frac{1}{4} e^{\frac{1}{n} S_{\alpha}} \text{Tr} (J_{\alpha} \gamma_{\alpha}) \quad (22)$$

$$\frac{dS_Y}{d\tau} = \frac{1}{4} \left(\sum_{\alpha=1}^K e^{\frac{1}{n} S_{\alpha}} \lambda_{\alpha} \right) \text{Tr} J_Y, \quad (23)$$

where J_Y and J_{α} are the quantum Fisher information matrices of the output and the α -th input, respectively.

The next step is to exploit the data processing inequality for the relative entropy between the state $\hat{\rho}_X$ of the input modes and its displaced version $\hat{D}_{\mathbf{x}} \hat{\rho}_X \hat{D}_{\mathbf{x}}^{\dagger}$, i.e.

$$S\left(\Phi(\hat{\rho}_X) \left\| \Phi\left(\hat{D}_{\mathbf{x}} \hat{\rho}_X \hat{D}_{\mathbf{x}}^{\dagger}\right)\right.\right) \leq S\left(\hat{\rho}_X \left\| \hat{D}_{\mathbf{x}} \hat{\rho}_X \hat{D}_{\mathbf{x}}^{\dagger}\right.\right), \quad (24)$$

where Φ is the CPTP map defined in (7). Another characterization of the latter can be obtained by exploiting the characteristic function representation of a quantum state $\hat{\rho}$, [28]

$$\chi(\mathbf{k}) := \text{Tr} \left(\hat{\rho} e^{i\mathbf{k}^T \hat{\mathbf{R}}} \right), \quad \hat{\rho} = \int \chi(\mathbf{k}) e^{-i\mathbf{k}^T \hat{\mathbf{R}}} \frac{d\mathbf{k}}{(2\pi)^n}. \quad (25)$$

Let then $\chi_X(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^{2Kn}$, and $\chi_Y(\mathbf{q})$, $\mathbf{q} \in \mathbb{R}^{2n}$ be the characteristic functions of the input and the output, respectively. From (7) and (8) we get

$$\chi_Y(\mathbf{q}) = \chi_X(M^T \mathbf{q}), \quad (26)$$

with M being the matrix entering in Eq. (8).

We then notice that displacing the inputs by \mathbf{x} , the output gets translated by

$$\mathbf{y} = M\mathbf{x} = \sum_{\alpha=1}^K M_{\alpha} \mathbf{x}_{\alpha},$$

i.e.

$$\hat{D}_{\mathbf{y}} \Phi(\hat{\rho}_X) \hat{D}_{\mathbf{y}}^{\dagger} = \Phi\left(\hat{D}_{\mathbf{x}} \hat{\rho}_X \hat{D}_{\mathbf{x}}^{\dagger}\right).$$

Therefore from (24) it follows

$$\begin{aligned} S\left(\Phi(\hat{\rho}_X) \left\| \hat{D}_{\mathbf{y}} \Phi(\hat{\rho}_X) \hat{D}_{\mathbf{y}}^{\dagger}\right.\right) &\leq S\left(\hat{\rho}_X \left\| \hat{D}_{\mathbf{x}} \hat{\rho}_X \hat{D}_{\mathbf{x}}^{\dagger}\right.\right) = \\ &= \sum_{\alpha=1}^K S\left(\hat{\rho}_{\alpha} \left\| \hat{D}_{\mathbf{x}_{\alpha}} \hat{\rho}_{\alpha} \hat{D}_{\mathbf{x}_{\alpha}}^{\dagger}\right.\right), \end{aligned} \quad (27)$$

where in the last passage we used the additivity of the relative entropy on product states. Since both the first and the last member of (27) are nonnegative and vanishing for $\mathbf{x} = 0$, inequality (27) translates to the Hessians. The variables are the x_{α}^i , $i = 1, \dots, 2n$, $\alpha = 1, \dots, K$, so the Hessian is a matrix with indices $(i, \alpha), (j, \beta)$, and the inequality reads

$$(M_{\alpha}^T J_Y M_{\beta})_{\alpha\beta} \leq (\delta_{\alpha\beta} J_{\alpha})_{\alpha\beta}, \quad (28)$$

where the indices i, j are left implicit. Finally, sandwiching (28) with $\lambda_{\alpha} e^{\frac{1}{n} S_{\alpha}} M_{\alpha}^{-T}$ on the left and its transpose $\lambda_{\beta} e^{\frac{1}{n} S_{\beta}} M_{\beta}^{-1}$ on the right (if M_{α} is not invertible, $\lambda_{\alpha} = 0$ and the corresponding terms are supposed to vanish), we get

$$\left(\sum_{\alpha=1}^K \lambda_{\alpha} e^{\frac{1}{n} S_{\alpha}} \right)^2 \text{tr} J_Y \leq \sum_{\alpha=1}^K \lambda_{\alpha} e^{\frac{2}{n} S_{\alpha}} \text{tr} (J_{\alpha} \gamma_{\alpha}), \quad (29)$$

and computing the parametric derivative of the rate (10) with (22) and (23), it is easy to show that (29) is equivalent to its positivity.

C. Asymptotic scaling

In this section we prove that the rate (10) tends to one for $\tau \rightarrow \infty$. For this purpose, we need the asymptotic scaling of the entropy under the dissipative evolution described by Eqs. (11), (12). Remember that we are evolving only the inputs with invertible M_α , for which $\gamma_\alpha > 0$.

1. A lower bound for the entropy

A lower bound for the entropy follows on expressing the state $\hat{\rho}$ in terms of its generalized Husimi function $Q_\Gamma(\mathbf{x})$, see e.g. Ref. [29]. Specifically, given a Gaussian state $\hat{\rho}_{\mathbf{x},\Gamma}$ characterized by first momentum $\mathbf{x} \in \mathbb{R}^{2n}$ and covariance matrix $\Gamma \geq \pm i\Delta$, we define

$$Q_\Gamma(\mathbf{x}) := \frac{\text{Tr}(\hat{\rho} \hat{\rho}_{\mathbf{x},\Gamma})}{(2\pi)^n} = \int e^{-\frac{1}{4}\mathbf{k}^T \Gamma \mathbf{k} - i\mathbf{k}^T \mathbf{x}} \chi(\mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^{2n}}, \quad (30)$$

where in the second line we used (25) and the fact that $e^{-\frac{1}{4}\mathbf{k}^T \Gamma \mathbf{k} + i\mathbf{k}^T \mathbf{x}}$ is the characteristic function of $\hat{\rho}_{\mathbf{x},\Gamma}$ (the conventional Husimi distribution [25] being recovered taking the states $\hat{\rho}_{\mathbf{x},\Gamma}$ to be displaced vacua, i.e. coherent states). By construction, $Q_\Gamma(\mathbf{x})$ is continuous in \mathbf{x} and positive: $Q_\Gamma(\mathbf{x}) \geq 0$. Furthermore since $\chi(0) = \text{Tr}\hat{\rho} = 1$ for any normalized state $\hat{\rho}$, we also have

$$\int Q_\Gamma(\mathbf{x}) d\mathbf{x} = 1 :$$

the generalized Husimi function $Q_\Gamma(\mathbf{x})$ is hence a probability distribution. Taking the Fourier transform of $Q_\Gamma(\mathbf{x})$, Eq. (30) can now be inverted obtaining

$$\hat{\rho} = \int Q_\Gamma(\mathbf{x}) \left(\int e^{\frac{1}{4}\mathbf{k}^T \Gamma \mathbf{k} + i\mathbf{k}^T \mathbf{x}} e^{-i\mathbf{k}^T \hat{\mathbf{R}}} \frac{d\mathbf{k}}{(2\pi)^n} \right) d\mathbf{x}. \quad (31)$$

Comparing with (25) the integral in parenthesis, it looks like a Gaussian “state” with covariance matrix $-\Gamma$ displaced by \mathbf{x} . Of course, this is not a well-defined object, and it makes sense only if integrated against smooth functions as $Q_\Gamma(\mathbf{x})$. However, if we formally define

$$\hat{\rho}_{-\Gamma} := \int e^{\frac{1}{4}\mathbf{k}^T \Gamma \mathbf{k}} e^{-i\mathbf{k}^T \hat{\mathbf{R}}} \frac{d\mathbf{k}}{(2\pi)^n},$$

Eq. (31) can be expressed as

$$\hat{\rho} = \int Q_\Gamma(\mathbf{x}) \hat{D}_\mathbf{x} \hat{\rho}_{-\Gamma} \hat{D}_\mathbf{x}^\dagger d\mathbf{x}.$$

Now we are ready to compute the lower bound for the entropy of a state evolved under a dissipative evolution defined as in Eqs. (11), (12). First we observe that even though the matrix γ entering (14) does not necessarily satisfy $\gamma \geq \pm i\Delta$ there exists always a constant $t_1 \geq 1$ such that $t_1\gamma$ fulfills such inequality, i.e. $t_1\gamma \geq \pm i\Delta$, the

existence of such t_1 being ensured by the positivity of γ . We can hence exploit the generalized Husimi representation (31) associated to the matrix $\Gamma = t_1\gamma$. For the linearity and the compatibility with translations (13) of the evolution (12), we can take the super-operator $e^{t\mathcal{L}_\gamma}$ that expresses the formal integration of the dissipative process (11) inside the integral:

$$e^{t\mathcal{L}_\gamma} \hat{\rho} = \int Q_{t_1\gamma}(\mathbf{x}) \hat{D}_\mathbf{x} (e^{t\mathcal{L}_\gamma} \hat{\rho}_{-t_1\gamma}) \hat{D}_\mathbf{x}^\dagger d\mathbf{x}, \quad (32)$$

and since $Q_{t_1\gamma}(\mathbf{x})$ is a probability distribution, the concavity of the von Neumann entropy implies $S(e^{t\mathcal{L}_\gamma} \hat{\rho}) \geq S(e^{t\mathcal{L}_\gamma} \hat{\rho}_{-t_1\gamma})$. The point now is that for $t > 2t_1$, $e^{t\mathcal{L}_\gamma} \hat{\rho}_{-t_1\gamma}$ is a Gaussian state with covariance matrix $(t - t_1)\gamma$, i.e. $e^{t\mathcal{L}_\gamma} \hat{\rho}_{-t_1\gamma} = \hat{\rho}_{(t-t_1)\gamma}$, and for $t \geq 2t_1$ it is a proper quantum state. Let ν_i , $i = 1, \dots, n$ be the symplectic eigenvalues of γ , i.e. the absolute values of the eigenvalues of $\gamma\Delta^{-1}$ [28]. Remembering that the entropy of the associated Gaussian state is

$$S(\hat{\rho}_\gamma) = \sum_{i=1}^n h(\nu_i),$$

where

$$h(\nu) = \frac{\nu+1}{2} \ln \frac{\nu+1}{2} - \frac{\nu-1}{2} \ln \frac{\nu-1}{2}, \quad (33)$$

we have

$$S(\hat{\rho}_{(t-t_1)\gamma}) = \sum_{i=1}^n h((t-t_1)\nu_i).$$

Since

$$h(\nu) = \ln \frac{\nu}{2} + 1 + \mathcal{O}\left(\frac{1}{\nu^2}\right) \quad \text{for } \nu \rightarrow \infty,$$

we finally get

$$\begin{aligned} S(e^{t\mathcal{L}_\gamma} \hat{\rho}) &\geq \sum_{i=1}^n \ln \frac{e(t-t_1)\nu_i}{2} + \mathcal{O}\left(\frac{1}{t^2}\right) = \\ &= n \ln \frac{et}{2} + \frac{1}{2} \ln \det \gamma + \mathcal{O}\left(\frac{1}{t}\right), \end{aligned} \quad (34)$$

where in the last step we have used that $\det \gamma = \prod_{i=1}^n \nu_i^2$.

2. An upper bound for the entropy

Given a state $\hat{\rho}$, let $\hat{\rho}_G$ be the Gaussian state with the same first and second moments. It is then possible to prove [30] that $S(\hat{\rho}_G) \geq S(\hat{\rho})$. Since the action of the evolution (12) on first and second moments is completely determined by them (and does not depend on other properties of the state), the Liouvillian \mathcal{L}_γ commutes with *Gaussianization*, i.e. $(e^{t\mathcal{L}_\gamma} \hat{\rho})_G = e^{t\mathcal{L}_\gamma} (\hat{\rho}_G)$, and we can

upper-bound the entropy of the evolved state with the one of the Gaussianized evolved state:

$$S(e^{t\mathcal{L}_\gamma}\hat{\rho}) \leq S((e^{t\mathcal{L}_\gamma}\hat{\rho})_G) = S(e^{t\mathcal{L}_\gamma}(\hat{\rho}_G)) . \quad (35)$$

From Eq. (14) we know that if σ is the covariance matrix of $\hat{\rho}$, the one of $e^{t\mathcal{L}_\gamma}\hat{\rho}$ is given by $\sigma+t\gamma$. Since the entropy does not depend on first moments, we have to compute the asymptotic behaviour of $S(\hat{\rho}_{\sigma+t\gamma})$. Let $t_2 > 0$ be such that $\sigma \leq t_2\gamma$. As $\gamma > 0$, such t_2 always exists: let λ_1^\downarrow be the biggest eigenvalue of σ , and $\mu_1^\uparrow > 0$ the smallest one of γ . Then $\sigma \leq \lambda_1^\downarrow \mathbb{1}_{2n} \leq \frac{\lambda_1^\downarrow}{\mu_1^\uparrow} \gamma$, so that $t = \frac{\lambda_1^\downarrow}{\mu_1^\uparrow}$ does the job. Now we remind that given two covariance matrices $\sigma' \leq \sigma''$, the Gaussian state $\hat{\rho}_{\sigma''}$ can be obtained applying an additive noise channel to $\hat{\rho}_{\sigma'}$. Since such channel is unital, it always increases the entropy, so we have $S(\hat{\rho}_{\sigma'}) \leq S(\hat{\rho}_{\sigma''})$. Applying this to $\sigma+t\gamma \leq (t_2+t)\gamma$, we get

$$\begin{aligned} S(\hat{\rho}_{\sigma+t\gamma}) &\leq S(\hat{\rho}_{(t_2+t)\gamma}) = \sum_{i=1}^n h((t_2+t)\nu_i) = \\ &= n \ln \frac{et}{2} + \frac{1}{2} \ln \det \gamma + \mathcal{O}\left(\frac{1}{t}\right) , \end{aligned} \quad (36)$$

where in the last step we have used that $\det \gamma = \prod_{i=1}^n \nu_i^2$.

3. Scaling of the rate

Putting together (34) and (36), we get

$$e^{\frac{1}{n}S(e^{t\mathcal{L}_\gamma}\hat{\rho})} = (\det \gamma)^{\frac{1}{2n}} \frac{et}{2} + \mathcal{O}(1) . \quad (37)$$

From section III A we can see that for our evolutions if M_α is invertible $\det \gamma_\alpha = 1$, so

$$e^{\frac{1}{n}S_\alpha(\tau)} = \frac{e}{2} t_\alpha(\tau) + \mathcal{O}(1)$$

and similarly

$$e^{\frac{1}{n}S_Y(\tau)} = \frac{e}{2} t_Y(\tau) + \mathcal{O}(1) .$$

Replacing this into the left-hand-side of Eq. (10), and remembering that if M_α is not invertible, then $\lambda_\alpha = 0$ and the corresponding terms vanish, from (16) and (17) it easily follows that such quantity tends to 1 in the $\tau \rightarrow \infty$ limit.

IV. CONCLUSIONS

The multi-mode version of the qEPI [22, 23] has been proposed and proved. This inequality, while probably not tight, provides a useful bound on the entropy production at the output of a multi-mode scattering process where independent collections of incoming, multi-mode, incoming inputs collide to produce a given output channel. Explicit examples of such a process are provided by broadband bosonic channels where the single signals are described as pulses propagating along optical fibers or in free space communication [26].

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